Programming AMMs on top of UniV3
[preprint]

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Contents

1 Introduction

Digital assets are becoming more and more popular, on-chain solutions are called to become way more efficient than current traditional market tools and provide new ways of building economic models.

Cryptocurrencies struggle from low liquidity and chicken and egg problem. Current systems use automated market maker (AMM) design to solve these problems. AMM design expects the users to provide the liquidity passively to the protocol incentivized by trading fees.

Recently a new approach was proposed in Uniswap V3 [2]. The defining idea of Uni V3 is the idea of concentrated liquidity: liquidity bounded within some price range. This feature makes it possible to increase the capital efficiency of AMM and allow liquidity providers (LPs) to approximate their preferred reserves curve while still being efficiently aggregated with the rest of the pool.

The purpose of this paper is to build a set of tools to create different market making strategies for liquidity providers by using concentrated liquidity feature or implementing different curves for payoff functions over UniV3. Also, we will show how designing the payoff function over UniV3 opens the door to implement in an efficient way any popular financial derivatives on-chain.

2 General Solution

We use definitions and some results from [3], [4].

2.1 Definitions

Trading function. A (path independent) CFMM is defined by its trading function $\psi : \mathbb{R}^2 \to \mathbb{R}$ and its reserves $R \in \mathbb{R}^2_+$. The reserve R_i specifies the quantity of coin i available to the CFMM contract, while the function ψ specifies the contract's behavior. More specifically, the contract will allow any agent to trade with the reserves, so long as the new reserves, $R' \in \mathbb{R}^2_+$, after the agent has added or withdrawn the required quantities, satisfy

$$\psi(R') \geq \psi(R).$$

Price vector. c is the price vector for the n coins the CFMM trades, such that c_i is the price of coin i in the external market.

Payoff function. Payoff is portfolio value of LP's capital in CFMM after arbitrage, represented by some function $V : \mathbb{R}^2_+ \to \mathbb{R}$. If CFMM is path-independent with constant k and its trading function ψ is concave and nonincreasing [4] shows that the function V is equal to

$$V(c) = \inf_{x \in \mathbb{R}^2_+} \{c^T x | \psi(x) \geq k\}.$$

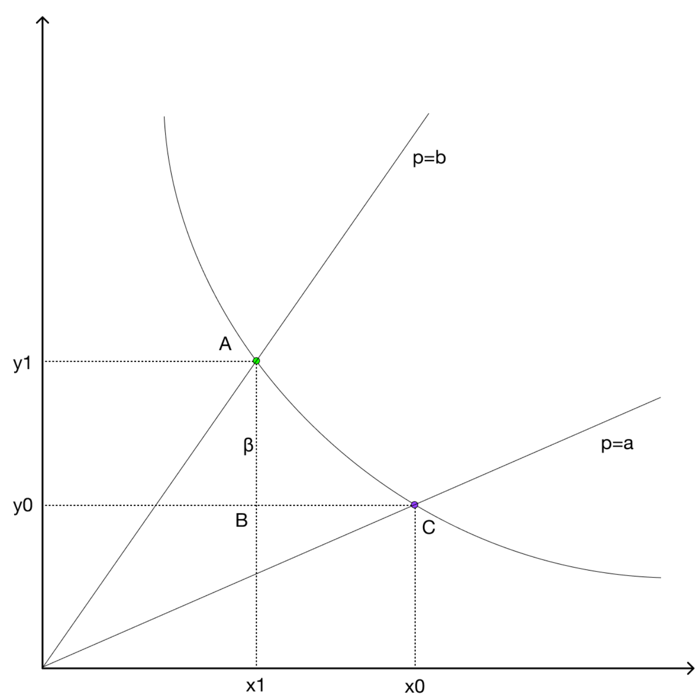
2.2 0-fees Uni V3 payoff function

Suppose the liquidity provider put liquidity on the price interval $c \in [a, b]$. Without loss of generality, assume that $c_2(t) = 1, \forall t$, $c_1(t) = c(t)$, and it's initial price $c(0) = a$. In this case, liquidity provider should provide only Token A (100% Token A, 0% Token B). Suppose LP has α Token A, 0 Token B at $c = a$, and β Token B, 0 Token A at $c = b$.

Then payoff function is equal to

$$V(c) = \begin{cases} \alpha c & c \leq a \\ \tilde{V}(c) & c \in (a, b) \\ \beta & c \geq b \end{cases}$$

So, we should find $\tilde{V}(c)$. Payoff function $\tilde{V}(c)$ changes according to CFMM invariant (see the graphic below).



$BC = \alpha, AB = \beta$ or $y_1 - y_0 = \beta, x_0 - x_1 = \alpha$.

Let's $p(0) = c \in [a, b]$. Our payoff is u. Let's x_v, y_v - virtual liquidity amount, x_r, y_r - real liquidity amount. Then we have system of equations:

$$\begin{cases} x_v y_v = L^2 \\ \frac{y_v}{x_v} = c \\ (x_r + \frac{L}{\sqrt{b}})(y_r + L\sqrt{a}) = L^2 \\ c x_r + y_r = u \\ c_r = \frac{y_r}{x_r} = \frac{\sqrt{c}-\sqrt{a}}{\frac{1}{\sqrt{c}}-\frac{1}{\sqrt{b}}} \end{cases}$$

The last equation can be derived from the fact that $dy_v/dx_v(x_v) = dy_r/dx_r(x_r)$

$$y_v^{'}(x) = -L^2/x_v^2 = -c$$

$$y_r = L(\frac{1}{x_r/L + 1/\sqrt{b}} - \sqrt{a})$$

$$y_r^{'} = -\frac{1}{(x_r/L + 1/\sqrt{b})^2} = -c$$

$$\frac{x_r}{L} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$$

$$\frac{y_r}{L} + \sqrt{a} = \sqrt{c}$$

Dividing last 2 equations we derive c_r

$$x_r = \frac{u}{c + c_r}$$

$$y_r = \frac{u c_r}{c + c_r}$$

$$\frac{u}{c + c_r} = L(\frac{1}{\sqrt{c}} - \frac{1}{\sqrt{b}})$$

$$L = \frac{u}{c(1/\sqrt{c} - 1/\sqrt{b}) + \sqrt{c} - \sqrt{a}} = \frac{u}{2\sqrt{c} - \sqrt{a} - c/\sqrt{b}}$$

Now we can find payoff function as

$$\tilde{V}(c, c_0) = x_r(c)c + y_r(c) = u\frac{2\sqrt{c} - \sqrt{a} - \frac{c}{\sqrt{b}}}{2\sqrt{c_0} - \sqrt{a} - \frac{c_0}{\sqrt{b}}}.$$

$$\tilde{V}(a, c_0) = u\frac{\sqrt{a} - \frac{a}{\sqrt{b}}}{2\sqrt{c_0} - \sqrt{a} - \frac{c_0}{\sqrt{b}}}$$

$$\tilde{V}(b, c_0) = u\frac{\sqrt{b} - \sqrt{a}}{2\sqrt{c_0} - \sqrt{a} - \frac{c_0}{\sqrt{b}}}$$

If liquidity is provided to $(0, \infty)$

$$W(c, c_0) = \lim_{a\to 0} \lim_{b\to +\infty} \left(u \frac{2\sqrt{c} - \sqrt{a} - \frac{c}{\sqrt{b}}}{2\sqrt{c_0} - \sqrt{a} - \frac{c_0}{\sqrt{b}}} \right) = u\sqrt{\frac{c}{c_0}},$$

which is equal to payoff function of Uniswap V2.

Now we can generalize this solution to any c_0:

- if $c_0 \le a$:

$$V(c) = \begin{cases} uc/c_0 & c \le a \\ \frac{ua}{c_0}\tilde{V}(c, a) & c \in (a, b) \\ \frac{ua}{c_0}\tilde{V}(b, a) & c \ge b \end{cases}$$

- if $c_0 \ge b$:

$$V(c) = \begin{cases} c\tilde{V}(a, b) & c \le a \\ \tilde{V}(c, b) & c \in (a, b) \\ u & c \ge b \end{cases}$$

- if $c_0 \in (a, b)$:

$$V(c) = \begin{cases} \frac{c}{a}\tilde{V}(a, c_0) & c \le a \\ \tilde{V}(c, c_0) & c \in (a, b) \\ \tilde{V}(b, c_0) & c \ge b \end{cases}$$

$V(c|a, b, c_0, u)$ is non-decreasing, concave function. Besides $V(c|a, b, c_0, u) = uV(c|a, b, c_0, 1)$. So, instead of $V(c|a, b, c_0, u)$ we will use $V_0(c|a, b, c_0) = V(c|a, b, c_0, 1)$. Here V_0 means that this is payoff function for zero-fees case.

2.3 non 0-fees UniV3 payoff function

If you draw the payoff function from the previous section, it would look like a ladder... Its stair will go up depending on how long the price is in the LPing interval $c \in [a, b]$...

Let's assume the trading volume going through the pool is $g(c)$. Interest earned by user can be calculated as

$$dI = \frac{gfL_0}{L_0 + L_g}dt$$

(if and only if $c \in [a, b]$), where $L_g(c)$ is general liquidity's invariant value (without user's liquidity), L_0 is invariant for providing liquidity at interval $[a, b]$.

In this section we will consider that fees are not continuously added to the pool (as it works, for example, in Uni V2).

Let's suggest $g = g_0 \min(L_g + L_0, \tilde{L})$ (ratio of trading volume and AMM portfolio is constant). Then

$$F(t) = \int_0^t g_0 f \min(L_0, \frac{\tilde{L} L_0}{L_0 + L_g}) d\tau,$$

where $F(t)$ - total fees.

Let's $\min(b/a) = 1 + \Delta_m$, Δ_m represents minimum price change available for providing liquidity. It is possible to show that $\Delta \geq \frac{2}{\sqrt{c_0}(\tilde{L} - L_g)}$: let's $a = (1-\delta)c_0$, $b = (1+\phi)c_0$. Then $L_0 \leq \tilde{L} - L_g$ or

$$1 \leq (\tilde{L} - L_g)\left[2\sqrt{c_0} - \sqrt{a} + c_0/\sqrt{b}\right],$$

which is equal to

$$\frac{1}{\tilde{L} - L_g} \leq \sqrt{c_0}\left(2 - 1 + \delta/2 - (1 - \phi/2)\right).$$

Taking into account $\Delta = \delta + \phi$ we get

$$\Delta_m \geq \frac{2}{\sqrt{c_0}(\tilde{L} - L_g)}.$$

Let's we have limitation $\min(b/a) - 1 = \Delta > \Delta_m$. Then we get formula for fees

$$F(t) = \frac{g_0 f t}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}}.$$

We can generalize this formula now. Let's $c_0 \in [a, b]$. Let's $t_{i,0} = \max(\tau: c(t) \in [a, b]\ \forall t \in [t_{i-1,1}, \tau],\ \tau \leq T)$, $t_{i,1} = \max(\tau: c(t) \notin (a, b)\ \forall t \in [t_{i,0}, \tau],\ \tau \leq T)$, where $t_{01} = 0$ and $i = 1, 2, ..., n$. According to definition $c(t) \in [a, b]$ in $[t_{i-1,0}, t_{i0}]$, and $c(t) \notin [a, b]$ otherwise. Then

$$F(t) = \begin{cases} F(t_{i,0}), & c(t) \geq b, t \in [t_{i,0}, t_{i,1}], \\ F(t_{i-1,0}) + \frac{g_0 f (t - t_{i-1,1})}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}}, & t \in [t_{i-1,1}, t_{i,0}], \end{cases}$$

where $i = 1, 2, ..., n$.

As you can see we can calculate $F(T)$ now as

$$F(T) = \frac{g_0 f \sum_{i=1,.,n}(t_{i,0} - t_{i-1,1})}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}}$$

and payoff is

$$V(T) = V_0(T) + F(T).$$

3 Optimal liquidity providing for deterministic prices

Leveraged LPing with static interval $[a, b]$ is formulated in previous sections. In this section, we will formulate and solve problem of payoff maximization when user can choose interval for LP-ing once or several times.

3.1 Passive strategy

In this section we will find solution for a problem when liquidity provider can choose a and b parameters once and lives with this decision till the end of time horizon $t = T$. Let's look at several scenarios: exponential price growth (both up and down) and flat price.

Case $\mu > 0$.

Let's consider $c(t) = c_0 \exp(\mu t)$. We should find maximum $V(T)$ by a and b. This optimal control problem can be solved an solution is described below:

Theorem 3.1. *If price of asset is $c(t) = c_0 \exp(\mu t)$, $\mu > 0$, $T \geq \ln(1+\Delta)/\mu$ then maximum payoff function for liquidity provider on CPMM with $xy = k$ and concentrated liquidity feature is equal to*

$$V^*(T) = \begin{cases} \frac{1}{\sqrt{1+\Delta}} e^{\mu T} + \frac{g_0 f e^{\mu T/2}}{\mu \sqrt{c_0}} \frac{\ln(1+\Delta)}{\sqrt{1+\Delta}-1} & \mu \leq \mu_*, \\ c_0 \exp(\mu T) & \mu > \mu_*. \end{cases} \tag{1}$$

Liquidity is provided on interval

$$[a_*, b_*] = \begin{cases} \left[\frac{c_0}{1+\Delta} e^{\mu T}, c_0 e^{\mu T}\right], & \mu \leq \mu_*, \\ \varnothing, & \mu > \mu_*, \end{cases} \tag{2}$$

where μ_* in equation above is solution of

$$\frac{g_0 f}{\mu \sqrt{c_0}} e^{-\mu T/2} = \frac{(\sqrt{1+\Delta}-1)^2}{\ln(1+\Delta)\sqrt{1+\Delta}}.$$

Proof: Proof is provided in Appendix.

Case $\mu < 0$. Let's swap prices: before we priced Token A in Token B, now will price Token B in Token A. That means price of Token B in Token A grows exponentially with parameter μ. So we get previous case.

Case $\mu = 0$.

Theorem 3.2. *If asset's prices is constant $c(t) = c_0$ then maximum payoff function for liquidity provider on CPMM with $xy = k$ and concentrated liquidity feature is equal to*

$$V^*(T) = c_0 + \frac{g_0 f}{\mu \sqrt{c_0} \sqrt{1+\Delta}} \frac{T\sqrt{1+\Delta}}{\sqrt{1+\Delta}-1} \tag{3}$$

Liquidity is provided on interval $[a_*, b_*] = [c_0, c_0 + \Delta]$.

Proof: This is trivial case.

We can get some conclusions from solutions above:

- liquidity provider should follow the trend: he should keep asset which price is going up as much as possible
- if trend is not strong enough, he should provide liquidity at the end of trend with maximum possible leverage.

3.2 Active strategy

Let's look at the situation from previous sections – is it better to provide liquidity at different intervals on different time frames?

Let's split time $[0, T]$ into K parts - $[kT/K, (k+1)T/K]$, where $k = 0, 1, ..., K-1$. We provide capital on k-th interval to $[c_0 \exp(\mu kT/K), c_0 \exp(\mu(k+1)T/K)]$.

Suppose re-providing liquidity cost is constant and equals q. Let's $V_p(t, T)$ - is payoff function for passive strategy from previous section, $V_a(t, T)$ - payoff function for active strategy. Then our goal is to maximize

$$V_a(T, T) - V_p(T, T) \to \max_K,$$

On interval $[0, T/K]$ $V_a(t, T) = V_p(t, T/K)$, on interval $[T/K, 2T/K]$

$$V_a(t, T) = (V_a(T/K, T) - q)V_a(t - T/K, T/K)$$

Then

$$V_a((i+1)T/K) = \left(V_a(\frac{iT}{K}, T) - q\right) V_a(t - \frac{iT}{K}, T),$$

where $i = 0, ..., K-1$.

Our goal is to maximize

$$V_a(T) - V_p(T) \to \max_K,$$

which is equivalent to

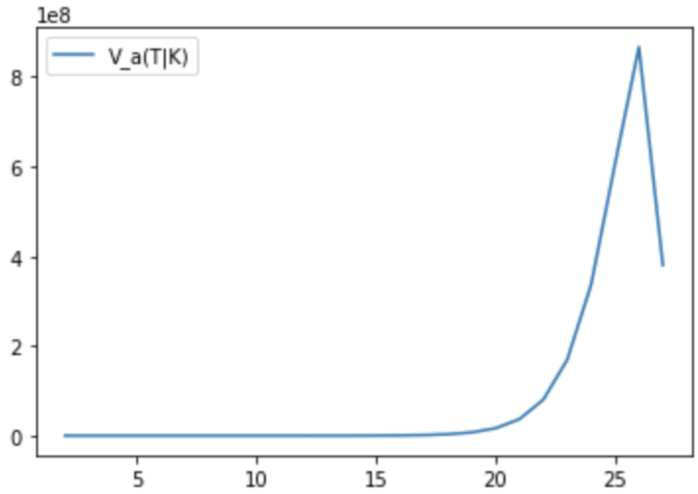
$$V_p^K(\frac{T}{K}, \frac{T}{K}) - q \sum_{j=0,..,K-1} V_p^j(\frac{T}{K}, \frac{T}{K}) \to \max_K,$$

or

$$V_p^K(\frac{T}{K}, \frac{T}{K}) - q\frac{V_p^K(\frac{T}{K}, \frac{T}{K}) - 1}{V_p(\frac{T}{K}, \frac{T}{K}) - 1} \to \max_{K \in \tilde{\mathbb{K}}},$$

where $\tilde{\mathbb{K}} = \{n \in \mathbb{N}, n \leq \mu T/\ln(1 + \Delta)\}$.

This maximization problem can be solved using numerical methods. Below is example of payoff function for active strategy for different K.



4 Liquidity management for stochastic price process

4.1 Expected payoff with zero fees

First we want to understand what is the evolution of the liquidity position value over time disregarding fees accumulation. Let's say we have one unit of token B that we invest into the pool at time t_0 and price c_0 to the interval (a, b). We want to derive the expected value of the position nominated in token B at arbitrary time t in the future.

As derived in (2.2):

$$L(a, b, c_0) = \frac{u}{c_0(1/\sqrt{c_0 \vee a \wedge b} - 1/\sqrt{b}) + \sqrt{c_0 \vee a \wedge b} - \sqrt{a}}$$

$$V^0_{a,b}(c, c_0) = L(c(1/\sqrt{c \vee a \wedge b} - 1/\sqrt{b}) + \sqrt{c \vee a \wedge b} - \sqrt{a}) \tag{4}$$

Thus the payoff at time t is

$$V_{a,b}(t) = L(a, b, c_0) \cdot \begin{cases} c_t \cdot \frac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}} & c_t \leq a \\ 2\sqrt{c_t} - \frac{c_t}{\sqrt{b}} - \sqrt{a} & c_t \in (a, b) \\ \sqrt{b} - \sqrt{a} & c_t \geq b \end{cases} \tag{5}$$

Now let's assume that c_t follows a Wiener process with parameters μ and σ

$$\frac{dc_t}{c_t} = \mu dt + \sigma dW_t, \text{where } W_t \text{ is a standard Wiener process} \tag{6}$$

Now cosider log-price process $S_t = \ln c_t - \ln c_0$. From the Itô's Lemma it follows that

$$dS_t = \tilde{\mu} dt + \sigma d\tilde{W}_t,\ \tilde{\mu} = \mu - \frac{\sigma^2}{2},\ S_0 = 1 \tag{7}$$

If we substitute S_t to (5):

$$V_{a,b}(t) = L(a,b,c_0) \cdot \begin{cases} c_0 \frac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}} \cdot e^{S_t} & S_t \leq \tilde{a} \\ 2\sqrt{c_0} e^{S_t/2} - \frac{c_0}{\sqrt{b}} e^{S_t} - \sqrt{a} & S_t \in (\tilde{a}, \tilde{b}) \\ \sqrt{b} - \sqrt{a} & S_t \geq \tilde{b} \end{cases} \tag{8}$$

$$\tilde{a} = \ln a - \ln c_0, \tilde{b} = \ln b - \ln c_0 \tag{9}$$

Lemma 4.1.

$$\mathbb{E}[e^{S_t/d} \mathbf{1}_{S_t \leq x}] = e^{\frac{t}{d}(\mu - \sigma^2/2 \cdot (1-1/d))} \Phi(\frac{x - \mu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}(\frac{1}{2} - \frac{1}{d}))$$

Proof.

$$\begin{aligned} \mathbb{E}[e^{S_t/d} \mathbf{1}_{S_t \leq x}] &= \mathbb{E}[e^{(\tilde{\mu} t + \sigma\sqrt{t}\mathsf{N})/d} \mathbf{1}_{\mu t + \sigma\sqrt{t}\mathsf{N} \leq x}] \\ &= e^{\tilde{\mu} t/d} \mathbb{E}[e^{\mathsf{N}/\hat{d}} \mathbf{1}_{\mathsf{N} \leq \hat{x}}] \end{aligned}$$

$$\hat{d} = \frac{d}{\sigma\sqrt{t}}, \hat{x} = \frac{x - \tilde{\mu} t}{\sigma\sqrt{t}}$$

$$\begin{aligned} \mathbb{E}[e^{\mathsf{N}/\hat{d}} \mathbf{1}_{\mathsf{N} \leq \hat{x}}] &= \int_{-\infty}^{\hat{x}} e^{x/\hat{d}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= e^{1/2\hat{d}^2} \int_{-\infty}^{\hat{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1/\hat{d})^2}{2}} dx \\ &= e^{1/2\hat{d}^2} \Phi(\hat{x} - 1/\hat{d}) \\ &= e^{t\sigma^2/2d^2} \Phi(\frac{x - \tilde{\mu} t}{\sigma\sqrt{t}} - \sigma\sqrt{t}/d) \end{aligned}$$

□

Lemma 4.2.

$$\mathbb{E}V_{a,b}(t) = L(a,b,c_0)\cdot[c_0e^{\mu t}\cdot\frac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}}\cdot\Phi(\frac{\ln a-\ln c_0-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}/2)+$$
$$+2\sqrt{c_0}e^{t(\mu/2-\sigma^2/8)}(\Phi(\frac{\ln b-\ln c_0-\mu t}{\sigma\sqrt{t}})-\Phi(\frac{\ln a-\ln c_0-\mu t}{\sigma\sqrt{t}}))-$$
$$-\frac{c_0e^{\mu t}}{\sqrt{b}}(\Phi(\frac{\ln b-\ln c_0-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}/2)-\Phi(\frac{\ln a-\ln c_0-\mu t}{\sigma\sqrt{t}}-\sigma\sqrt{t}/2))+$$
$$-\sqrt{a}(\Phi(\frac{\ln b-\ln c_0-\mu t}{\sigma\sqrt{t}}+\sigma\sqrt{t}/2)-\Phi(\frac{\ln a-\ln c_0-\mu t}{\sigma\sqrt{t}}+\sigma\sqrt{t}/2))+$$
$$+(\sqrt{b}-\sqrt{a})(1-\Phi(\frac{\ln b-\ln c_0-\mu t}{\sigma\sqrt{t}}+\sigma\sqrt{t}/2))]$$

Proof. Follows from Lemma 4.1 and (8): □

4.2 Expected payoff with fees

Now let's derive $V^F_{a,b}(t)$ - the expected payoff of liquidity position including trading fees.

$$V^F_{a,b}(t) = V_{a,b}(t) + F_{a,b}(t) \tag{10}$$

Note that trading fees are generated only when price $c_t \in (a,b)$. Let's define $\tau^c_{a,b}(T)$ - the amount of time that price $c_t \in (a,b)$ on the time interval $(0,T)$.

$$\tau^c_{a,b}(T) = \int_0^T \mathbf{1}_{c_t\in(a,b)}dt \tag{11}$$

Using the value of L in (2.2). Recall that g_0 and f are constants. Then

$$F_{a,b}(T) = L(a,b,c_0)\cdot g_0\cdot f\cdot\tau^c_{a,b}(T) \tag{12}$$

We already know all parts of the equation except $\tau^c_{a,b}(T)$

Lemma 4.3. *Let* $\tau^S_{a,b}(T) = \int_0^T \mathbf{1}_{S_t\in(a,b)}dt$. *And* $\tilde{a}$ *and* $\tilde{b}$ *are defined in* (9). *Then*

$$\tau^S_{\tilde{a},\tilde{b}}(T) = \tau^c_{a,b}(T) \tag{13}$$

Proof. $S_t \le \tilde{a} \leftrightarrow c_t \le a$ □

Lemma 4.4.

$$\mathbb{E}\tau^S_{a,b}(T) = \mathbb{E}\tau^S_a(T) - \mathbb{E}\tau^S_b(T) \tag{14}$$

Lemma 4.5.

$$\mathbb{E}\tau^S_a(T) = T - \int_0^T \Phi(\frac{a-\tilde{\mu}t}{\sigma\sqrt{t}}) \tag{15}$$

where Φ *is CDF of standard normal distribution*

Proof.

$$\mathbb{E}\tau_a^S(T) = \int_0^T \mathbb{P}[S_t \geq a]dt =$$
$$= T - \int_0^T \Phi(\frac{a-\tilde{\mu}t}{\sigma\sqrt{t}})dt$$

□

From the lemmas above it follows that

$$\mathbb{E}[F_{a,b}(T)] = L(a,b,c_0)\cdot g_0 \cdot f \cdot \int_0^T \Phi(\frac{\ln b - \ln c_0 - \mu t}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2})dt - \tag{16}$$
$$- L(a,b,c_0)\cdot g_0 \cdot f \cdot \int_0^T \Phi(\frac{\ln a - \ln c_0 - \mu t}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2})dt$$

Lemma 4.6.

$$\mathbb{E}V_{a,b}(t) = L(a,b,c_0)\cdot[c_0 e^{\mu t}\cdot\frac{\sqrt{b}-\sqrt{a}}{\sqrt{ab}}\cdot\Phi(\frac{\ln a - \ln c_0 - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}/2)+$$
$$+ 2\sqrt{c_0}e^{t(\mu/2-\sigma^2/8)}(\Phi(\frac{\ln b - \ln c_0 - \mu t}{\sigma\sqrt{t}}) - \Phi(\frac{\ln a - \ln c_0 - \mu t}{\sigma\sqrt{t}}))-$$
$$- \frac{c_0 e^{\mu t}}{\sqrt{b}}(\Phi(\frac{\ln b - \ln c_0 - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}/2) - \Phi(\frac{\ln a - \ln c_0 - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}/2))+$$
$$- \sqrt{a}(\Phi(\frac{\ln b - \ln c_0 - \mu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}/2) - \Phi(\frac{\ln a - \ln c_0 - \mu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}/2))+$$
$$+ (\sqrt{b}-\sqrt{a})(1-\Phi(\frac{\ln b - \ln c_0 - \mu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}/2))]+$$
$$+ L(a,b,c_0)\cdot g_0\cdot f\cdot\int_0^t \Phi(\frac{\ln b - \ln c_0 - \mu\hat{t}}{\sigma\sqrt{\hat{t}}} + \frac{\sigma\sqrt{\hat{t}}}{2})d\hat{t} -$$
$$- L(a,b,c_0)\cdot g_0\cdot f\cdot\int_0^t \Phi(\frac{\ln a - \ln c_0 - \mu\hat{t}}{\sigma\sqrt{\hat{t}}} + \frac{\sigma\sqrt{\hat{t}}}{2})d\hat{t}$$

4.3 Optimal greedy strategy

Let's see how we can build a greedy strategy for managing liquidity. Let's divide time into n equal periods, e.g. weekly or daily periods. The end of each interval is t_n. The length of the interval is based on the stability of μ and σ on that interval. At the end of each interval we estimate new μ_t and σ_t and solve solve the optimization problem

$$\mathbb{E}[V^F_{a,b}(T,t_n)] \to max_{a,b} \tag{17}$$

Since $\mathbb{E}[V^F_{a,b}(T,t_n)]$ is C^2 smooth and separable by a and b we can solve this problem using gradient descent family methods (e.g. stochastic gradient descent).

If each rebalance will cost $C(t_n)$ and a_n^* and b_n^* are optimal solutions then we do rebalance iff

$$\mathbb{E}[V^F_{a_n^*,b_n^*}(T,t_n)] - \mathbb{E}[V^F_{a_{n-1}^*,b_{n-1}^*}(T,t_n)] \geq C(t_n) \tag{18}$$

Thus we have a set of rules that will yield optimal investment return for long-term liquidity management on Uniswap V3.

It's worth noting that in general case greedy strategy is not the most effective strategy. However, the accuracy of forecast for μ and σ are declining for more distant future. Therefore the best we can do is use the greedy strategy for the upcoming period.

5 Payoff approximation for AMMs

5.1 Balancer AMM

Balancer's payoff function for a two-sided market [3] is

$$V_B(c) = c_1^w c_2^{1-w}$$

Assume $c_1(t) = c(t)$, $c_2(t) = 1$ and we can get $V_B(c) = c^w$. The problem can be formulated as Is there exists series of Uniswap V3 payoff functions V_i that approximates Balancer's payoff function V_B:

$$V_B(c) = \sum_{i \in \mathbb{N}} V_i(c) + o(1).$$

We can show that it's possible. Suppose price at the beginning is $c_0 = a$. Let's provide liquidity to Uni V3 on interval $[a, b]$ to get behavior of pool like Balancer's pool on price interval $[a, b]$. Let's divide $[a, b]$ to intervals $[a + sk, a + s(k+1)]$, where $k \in \mathbb{N}, k \leq I(b) = ceil((b-a)/s)$, s - minimum available interval on UniSwap V3. Then $V_B(c) = c^w$.

$$V_i(c) = \begin{cases} \frac{u_i c}{a} & c \leq a + si \\ \frac{u_i}{a}\tilde{V}(c, a + si) & c \in (a + si, a + s(i+1)) \\ \frac{u_i}{a}\tilde{V}(a + (i+1)s, a + si) & c \geq a + s(i+1) \end{cases}$$

$$V(c) = \left(\sum_{i \in \mathbb{N}, i \leq I(b)} V_i(c) \right)$$

We want to find such u_i, so that

$$V_B(a + s(i + 0.5)) = V(a + s(i + 0.5)) = (a + s(i + 0.5))^w / c_0^w, \ \forall i \in [0, I(b)].$$

We have system of $n = I(b) + 1$ equations, so:

$$Au = b,$$

where

$$a_{ij} = \begin{cases} \frac{u_i \tilde{V}(a+(i+1)s, a+si)}{a}, & 1 \leq i < j \leq n, \\ \frac{u_i \tilde{V}(c_j, a+si)}{a}, & 1 \leq i = j \leq n, \\ \frac{u_i c_j}{a}, & 1 \leq j < i \leq n, \end{cases}$$

where $c_j = a + s(j - 0.5)$, $b_j = (a + s(j - 0.5))^w, j = 1, ..., n - 1, b_n = 1$.

It is possible to show that this matrix equation has unique non-negative solution, but it is out of scope of this paper (determinant of matrix has non-zero value, all matrix elements are non-negative numbers and $A[0, .., 0]^T < b$).

This unique solution of matrix equation can be fined by Gaussian elimination algorithm [5] (this matrix should be transformed to trianlgular matrix).

5.2 Curve AMM

Approximation algorithm from Balancer's section can also be used to create Curve-like AMM (this algorithm can be used to generate AMM with any payoff function).

6 Building derivatives on Uni V3

Let's show how Uniswap V3 can be used for building custom derivatives market.

If we consider $s << 1$, and liquidity is provided to very small interval $[a, a+0]$, then we can find that payoff function we've got $V(c)$ is payoff for selling a-strike put-option $P(c, a)$ plus strike amount a.

According to put- and call- options arbitrage formula [9]

$$C(c) + K = P(c) + c.$$

Taking into account $V(c) = a - P(c, a)$, we get that $C(c) + V(c) = c$. So call option can be got as selling LP-token and holding 1 unit of Token A.

7 Further research

We want to create Reverse Engineering of AMMs: when you can set AMM's payoff function, and AMM uses it to define trading function... This makes it possible to create different AMMs for derivatives.

8 Conclusion

Uniswap V3 is coming soon. It will provide a flexible tool to control the AMM mechanism. Previously AMM settings were fixed and couldn't be changed. Now we're able to program the AMMs. Using this mechanism we're implementing different payoff functions to create a new approach for decentralized on-chain marketmaking strategies.

The first implementation will be for leveraging LP-ing and will go further with new financial instruments.

We believe that combining these tools with Uniswap and it's user base will augment the adoption of blockchain derivatives and will accelerate the era of transferring financial systems on-chain.

A Proof of Theorem3.1

There are 6 possible scenarios:

1. $a \le c_0$ $b \ge c_0 e^{\mu T}$
2. $c_0 \le a \le c_0 e^{\mu T} \le b$
3. $c_0 \le a < b \le c_0 e^{\mu T}$
4. $c_0 e^{\mu T} \le a \le b$
5. $a \le c_0 \le b \le c_0 e^{\mu T}$
6. $b \le c_0$

Let's look at each one separately and show that the optimal strategy is in case 3. It's easy to find that in 6-th case $V(T) = 1$, in 4-th case $V_4(T) = e^{\mu T}$.

Now let's look at 1-st case:

$$V(T) = \frac{2\sqrt{c_T} - \sqrt{a} - c_T/\sqrt{b}}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}} + \frac{g_0 f T}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}} =$$

$$= 1 + \frac{2(\sqrt{c_T} - \sqrt{c_0}) - \frac{c_T - c_0}{\sqrt{b}} + g_0 f T}{2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b}}.$$

$$V_a'(T) = \frac{2(\sqrt{c_T} - \sqrt{c_0}) - \frac{c_T - c_0}{\sqrt{b}} + g_0 f T}{(2\sqrt{c_0} - \sqrt{a} - c_0/\sqrt{b})^2} \frac{1}{2\sqrt{a}}.$$

Let's check the sign of numerator:

$$2(\sqrt{c_T} - \sqrt{c_0})\sqrt{b} - (c_T - c_0) + g_0 f T\sqrt{b} = g_0 f T\sqrt{b} + (\sqrt{b} - \sqrt{c_0})^2 - (\sqrt{b} - \sqrt{c_T})^2.$$

Taking into account that $\sqrt{b} - \sqrt{c_T} < \sqrt{b} - \sqrt{c_0}$ we get $V_a'(T) > 0$. That means $a_* \ge c_0$ and optimal solutions is not in 1-st case. Same computations will be true for 5-th case.

Now let's look at 2-nd case.

$$V(T) = \frac{a}{c_0}\left(\frac{2\sqrt{c_T} - \sqrt{a} - c_T/\sqrt{b} + \frac{g_0 f}{\mu}\ln(c_T/a)}{\sqrt{a} - a/\sqrt{b}}\right) =$$

$$\frac{a}{c_0}\frac{c_T}{a}\frac{\frac{2}{\sqrt{c_T}} - \frac{\sqrt{a}}{c_T} - \frac{1}{\sqrt{b}} + \frac{g_0 f}{\mu c_T}\ln(c_T/a)}{\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}} =$$

$$= \frac{c_T}{c_0}\left(1 + \frac{\frac{2}{\sqrt{c_T}} - \frac{\sqrt{a}}{c_T} - \frac{1}{\sqrt{a}} + \frac{g_0 f}{\mu c_T}\ln(c_T/a)}{\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}}\right)$$

Then

$$V_b'(T) = \frac{c_T}{c_0} \frac{\frac{2}{\sqrt{c_T}} - \frac{\sqrt{a}}{c_T} - \frac{1}{\sqrt{a}} + \frac{g_0 f}{\mu c_T}\ln(c_T/a)}{\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right)^2}\left(-\frac{1}{2b\sqrt{b}}\right)$$

Let's look numerator of second fraction:

$$\frac{2}{\sqrt{c_T}} - \frac{\sqrt{a}}{c_T} - \frac{1}{\sqrt{a}} + \frac{g_0 f}{\mu c_T}\ln(c_T/a) = q(a)$$

Take into account $f(c_T) = 0$. Let's look at its derivative function:

$$q'(a) = -\frac{1}{2\sqrt{a}c_T} + \frac{1}{2\sqrt{a}a} - \frac{g_0 f}{\mu c_T a} = \frac{c_T - a - 2g_0 f\sqrt{a}}{2c_T a\sqrt{a}}.$$

Numerator is decreasing function by a and $q'(c_T) < 0$. That means if $q(c_0) > 0$ then $f(a) \geq 0\ \forall a \in [c_0, c_T]$. In this case $b_* \leq c_T$ and solution is in 5 case. So, solution in 5th case if

$$g_0 f T \geq \sqrt{c_0}(\exp(\mu T/2) - 1)^2.$$

Otherwise there is exists $\hat{a} \in [c_0, c_T] : \ q(\hat{a}) = 0$. That means $V_b'(T) < 0$ if $a \in (\hat{a}, c_T]$, $V_b'(T) = 0$ if $a = \hat{a}$ and $V_b'(T) > 0$ if $a \in [c_0, \hat{a})$. Case $a_* \in (\hat{a}, c_T]$ is equivalent to 5th case, so let's look at possible solution on $a_* \in [c_0, \hat{a}]$. In this case payoff function is

$$V(T) = \frac{a}{c_0}\left(\frac{2\sqrt{c_T} - \sqrt{a} + \frac{g_0 f}{\mu}\ln(c_T/a)}{\sqrt{a}}\right) = \frac{\frac{g_0 f\sqrt{a}}{\mu}\ln(c_T/a) + 2\sqrt{ac_T} - a}{c_0}.$$

Let's $a = c_T\exp(-\mu\sigma)$, $\sigma \in [0, T]$. Then

$$(\frac{c_0}{c_T}V(T)) = \frac{g_0 f}{\sqrt{c_T}}e^{-\mu\sigma/2}\sigma + 2e^{-\mu\sigma/2} - e^{-\mu\sigma}$$

Then it's derivative function is

$$e^{-\mu\sigma/2}\left(\frac{g_0 f}{\sqrt{c_T}} - \frac{g_0 f\mu\sigma}{2\sqrt{c_T}} - \mu(1 - e^{-\mu\sigma/2})\right) = w(\sigma)e^{-\mu\sigma/2}$$

$$w' = -\frac{g_0 f\mu}{2\sqrt{c_T}} - \frac{\mu^2}{2}e^{-\mu\sigma/2} < 0,\ w(0) > 0.$$

So, there is two possible scenarios depending on parameters value:

$$\sigma_* = argmax V(a) = \begin{cases} T, & \text{if } g_0 f(1 - \mu T/2) - \mu\sqrt{c_0}(e^{\mu T/2} - 1) > 0 \\ \sigma_1, & \text{otherwise,} \end{cases}$$

where σ_1 is solution of equation

$$g_0 f(1-\mu\sigma/2) - \mu\sqrt{c_0}e^{\mu T/2}(1-e^{-\mu\sigma/2}) = 0.$$

Then $a_* = \min\{\hat{a}, c_0 \exp\left(\mu(T-\sigma_*)\right)\}$.

Now we can formulate solution for 2nd case. Let's $\xi = g_0 fT - \sqrt{c_0}(1-\exp(\mu T/2))^2$ and $\lambda = g_0 f(1-\mu T/2) - \mu\sqrt{c_0}(e^{\mu T/2}-1)$. Then

$$V_2(T) = \begin{cases} \frac{g_0 f\sqrt{a_*}}{c_0\mu}\ln(c_T/a_*) + 2\sqrt{a_* c_T/c_0^2} - a_*/c_0, & \text{if } \xi < 0 \text{ and } \lambda \le 0, \\ \tilde{V}(c_T|a = c_0, b = +\infty), & \text{if } \xi < 0 \text{ and } \lambda > 0, \\ V_5(T), & \text{if } \xi \ge 0 \end{cases}$$

Now let's look at 3-rd case.

$$V(T) = \frac{\sqrt{ab}}{c_0}\left(1 + \frac{g_0 f}{\mu}\frac{\ln b - \ln a}{\sqrt{b}-\sqrt{a}}\right).$$

Let's make change of variable $u = \sqrt{b/a}$. Then $u^2 \ge 1+\Delta$, $a \in [c_0, \frac{c_0 e^{\mu T}}{u^2}]$. Then we have

$$V(T) = \frac{ua}{c_0}\left(1 + \frac{2g_0 f}{\mu\sqrt{a}}\frac{\ln u}{u-1}\right) = f(u,a)$$

Let's research behave of this function:

$$f_a' = \frac{u}{c_0}\left(1 + \frac{2g_0 f}{\mu\sqrt{a}}\frac{\ln u}{u-1}\right).$$

We have $u \ge \sqrt{1+\Delta}$, so $f_a' > 0$. That means that we should research behave of function when $a = \frac{c_T}{u^2}$:

$$f = \frac{c_T}{u} + \frac{2g_0 f\sqrt{c_T}}{\mu}\frac{\ln u}{u-1}.$$

Let's research behave of second part:

$$U(u) = \left(\frac{\ln u}{u-1}\right)' = \frac{1 - 1/u - \ln u}{(u-1)^2}$$

.

Numenator is equal to 0 if $u = 1$ and it's decreasing function on $u \ge 1$. That means $U(u) \le 0$ on $u \ge \sqrt{1+\Delta}$. So $f(u)$ is sum of two decreasing functions. Therefore $u_* = \sqrt{1+\Delta}$, $a_* = c_T/(1+\Delta)$. Substitute this in equation for $V(T)$ we get

$$V_*(T) = \frac{c_T}{c_0\sqrt{1+\Delta}} + \frac{g_0 f\sqrt{c_T}}{\mu c_0}\frac{\ln(1+\Delta)}{\sqrt{1+\Delta}-1}$$

Now we should compare this value with value from 2-nd and 4-th. $V(T) = \max\{V_2(T), V_3(T), V_5(T)\}$.

Let's compare $V_2(T)$ and $V_5(T)$. If $\xi < 0$ and $\lambda > 0$ then we should compare

$$\frac{g_0 f T}{\sqrt{c_0}} + 2e^{\mu T/2} - 1 \vee e^{\mu T},$$

which is equal to

$$\frac{g_0 f T}{\sqrt{c_0}} - \left(e^{\mu T/2} - 1\right)^2 \vee 0.$$

This equation ≤ 0 by the definition of ξ. So $V_2(T) \leq V_5(T)$ for $\xi < 0$ and $\lambda > 0$.

Now let's compare $V_2(T)$ and $V_5(T)$ for $\xi < 0$ and $\lambda \leq 0$:

$$\frac{g_0 f \sqrt{a_*}}{c_0 \mu} \ln(c_T/a_*) + 2\sqrt{a_* c_T/c_0^2} - a_*/c_0 \vee e^{\mu T}$$

Let's $a_* = c_T e^{\mu\sigma}$. Then inequality above is equal to

$$\frac{g_0 f \sqrt{a_*}}{\mu} \ln(c_T/a_*) + 2\sqrt{a_* c_T} - a_* - c_0 e^{\mu T} \vee 0$$

or

$$q(a_*) c_T \sqrt{a_*} \vee 0.$$

According to definition $a_* \leq \hat{a}$, where $q(\hat{a}) = 0$. Taking into account that $q(a_*) \leq q(\hat{a}) = 0$ (look at analysis f $q(\cdot)$ above) we get $V_2(T) \leq V_5(T)$ for this case also. Then we get $V_2(T) \leq V_5(T)$ for all cases. That means our maximum either on 3-th case or 5-th case. Comparing $V(T)$ for this both cases we can find that if $\mu > \mu_*$ then payoff function on 4-5h case is higher and vica versa.

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